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External symmetry in general relativity

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Abstract. We propose a generalization of the isometry transformations to the geometric context of the spin field theories where the local frames are explicitly involved. We define the external symmetry transformations as isometries combined with suitable tetrad gauge transformations and we show that they form a group which is locally isomorphic with the isometry one. We point out that these symmetry transformations leave invariant the field equations in local frames and have generators with specific spin terms that represent new physical observables. The examples we give are the generators of the central symmetry and those of the maximal symmetries of the de Sitter and anti-de Sitter spacetimes for which we derive the spin terms in different tetrad gauge fixings.

1. Introduction

In general relativity [1–3] the development of the quantum field theory in curved spacetimes [4] gives rise to many difficult problems related to the physical interpretation of the one-particle quantum modes that may indicate how to quantize the field. This is because the form and the properties of the particular solutions of the free field equations, in the cases when these can be analytically solved [5, 6], are strongly dependent on the procedure of separation of variables and, implicitly, on the choice of the local chart. Moreover, when the fields have spin the situation is more complicated since then the field equations and, therefore, the form of their particular solutions depend, in addition, on the tetrad gauge in which one works [1, 7]. In these conditions it would be helpful to use the traditional method of the quantum theory in flat spacetime based on the complete sets of commuting operators that determine the quantum modes as common eigenstates and give physical meaning to the constants of the separation of variables which are just the eigenvalues of these operators. A good step in this direction could be to proceed as in special relativity, looking for the generators of the geometric symmetries similar to the familiar momentum, angular momentum and spin operators of the Poincaré covariant field theories [8].

However, the relativistic covariance in the sense of general relativity is too general to play the same role as the Lorentz or Poincaré covariance in special relativity. In other respects, the tetrad gauge covariance of the theories with spin represents another kind of general symmetry that is not able to itself produce conserved observables [1]. For this reason we have to concentrate only upon some special transformations of a well defined Lie group with significant parametrization from the geometric point of view, which should leave invariant the form of the field equations. Obviously, these may be just the isometry transformations that point out the spacetime symmetry giving us the specific Killing vectors [1, 3, 9]. The physical fields, (minimally) coupled with the gravitational one, take over this symmetry, transforming according to appropriate representations of the isometry group. In the case of the scalar vector or tensor fields these representations are completely defined by the well known rules of the general coordinate transformations since the isometries are in fact particular automorphisms. The problem of the behaviour under isometries of the fields with half-integer spin that explicitly depend on the tetrad gauge fixing remains open.

Another important problem is how to define the generators of these representations for any spin. It is known that there is a standard operator-valued representation of the isometry group in the space of scalar functions whose generators can be written with the help of the Killing vectors in a similar manner as are the orbital angular momentum operators of special relativity. But how can we define the corresponding spin parts of the generators of the representations according which the fields with spin may transform? In the case of the Dirac field these spin parts are known for the angular momentum in the diagonal tetrad gauge of central backgrounds [10] and even for any generator corresponding to a Killing vector in any chart and arbitrary tetrad gauge fixing [11]. However, we cannot say that this problem is then generally solved for fields with any spin obeying different free or coupled field equations.

Our aim here is to propose a way to solve the above-mentioned problems in the case of the tetrad gauge covariant theories of fields defined on curved spacetimes with given symmetries. Our main objectives are to find how we must transform these fields under isometries in order to leave the form of the field equations invariant and to derive the general expression of the generators of these transformations.

We start with the idea that if we intend to study the symmetry of a physical theory we must take into consideration the whole geometric context, including the positions of the local frames given by the tetrad fields. This is because the spin is defined just with respect to the axes of these frames. Then it is natural to require the symmetry transformations to preserve not only the form of the metric tensor but the tetrad gauge too. Such transformations can be constructed as isometries combined with suitable tetrad gauge transformations necessary to keep the tetrad field components unchanged. In this way we obtain the *external symmetry* group, showing that it is locally isomorphic with the isometry group. Moreover, there are arguments that this is in fact isomorphic with the universal covering group of the isometry one.

The next step is to define the operator-valued representations of the external symmetry group carried by spaces of fields with spin. We point out that these are induced by the linear finite-dimensional representations of the $SL(2, \mathbb{C})$ group. This is why the symmetry transformations which leave the field equations invariant have generators with a composite structure. These have the usual orbital terms of the scalar representation and, in addition, specific spin terms which depend on the choice of the tetrad gauge, even in the case of the fields with integer spin. In general, the spin and orbital terms do not commute to each other apart from some special gauge fixings where the fields transform manifestly covariant under external symmetry transformations.

Based on these results, we study two important examples, namely the central symmetry and the maximal symmetry of the de Sitter (dS) and anti-de Sitter (AdS) spacetimes. In the case of the central geometries we use central charts with Cartesian coordinates and the Cartesian tetrad gauge which allowed us recently to find new analytical solutions of the Dirac equation [12]. We show that in this gauge fixing the central symmetry becomes global and, consequently, the spin parts of its generators are the same as those of special relativity [13, 14]. This is important from the technical point of view since in the largely used diagonal tetrad gauge in spherical coordinates [15, 16] one obtains that the spin terms are partially hidden [10]. For the dS and AdS spacetimes we calculate the generators of the representations of the external symmetry group in central charts with our Cartesian gauge and in Minkowskian charts [1] with another gauge where the fields are manifestly covariant under the Lorentz symmetry [17].

In section 2 we point out that the general relativistic covariance and the tetrad gauge one can be treated together, introducing the group of combined transformations defined as tetrad gauge transformations, followed by automorphisms. Section 3 is devoted to our approach. Therein we define the external symmetry transformations, show that these form a Lie group and study the operator-valued representations of this group and its Lie algebra. In sections 4 and 5 we discuss the mentioned examples.

We present our proposal in terms of the relativistic quantum mechanics in the sense of general relativity, avoiding consideration of the specific problems of the quantum field theory or use of too complicated mathematical methods. We work in natural units with $\hbar = c = 1$.

2. Relativistic covariance

In the Lagrangian field theory in curved spacetimes the relativistic covariant equations of scalar, vector or tensor fields arise from actions that are invariant under general coordinate transformations. Moreover, when the fields have spin in the sense of the $SL(2,\mathbb{C})$ symmetry then the action must be invariant under tetrad gauge transformations [7]. The first step to our approach we propose here is to embed both these kinds of transformations into new ones, called combined transformations, to help us to understand the relativistic covariance in its most general terms.

2.1. Gauge transformations

Let us consider the curved spacetime M and a local chart (natural frame) of coordinates x^μ, $\mu = 0, 1, 2, 3$. Given a gauge, we denote by $e_{\hat\mu}(x)$ the tetrad fields that define the local frames, in each point x, and by $\hat e^{\hat\mu}(x)$ those of the corresponding coframes. These have the usual orthonormalization properties

$$\hat e^{\hat\mu}_\alpha\, e^\alpha_{\hat\nu} = \delta^{\hat\mu}_{\hat\nu} \qquad \hat e^{\hat\mu}_\alpha\, e^\beta_{\hat\mu} = \delta^\beta_\alpha \qquad e_{\hat\mu}\cdot e_{\hat\nu} = \eta_{\hat\mu\hat\nu} \qquad \hat e^{\hat\mu}\cdot\hat e^{\hat\nu} = \eta^{\hat\mu\hat\nu} \tag{1}$$

where $\eta = \mathrm{diag}(1,-1,-1,-1)$ is the Minkowski metric. From the line element

$$\mathrm{d}s^2 = \eta_{\hat\mu\hat\nu}\,\mathrm{d}\hat x^{\hat\mu}\,\mathrm{d}\hat x^{\hat\nu} = g_{\mu\nu}(x)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \tag{2}$$

expressed in terms of 1-forms, $\mathrm{d}\hat x^{\hat\mu} = \hat e^{\hat\mu}_\nu\,\mathrm{d}x^\nu$, we get the components of the metric tensor of the natural frame,

$$g_{\mu\nu} = \eta_{\hat\alpha\hat\beta}\hat e^{\hat\alpha}_\mu \hat e^{\hat\beta}_\nu \qquad g^{\mu\nu} = \eta^{\hat\alpha\hat\beta} e^\mu_{\hat\alpha} e^\nu_{\hat\beta}. \tag{3}$$

These raise or lower the *natural* vector indices, i.e. the Greek symbols, ranging from zero to three, while for the *local* vector indices, denoted by hatted Greek symbols and having the same range, we must use the Minkowski metric. The derivatives $\hat\partial_{\hat\nu} = e^\mu_{\hat\nu}\partial_\mu$ satisfy the commutation rules

$$[\hat\partial_{\hat\mu}, \hat\partial_{\hat\nu}] = e^\alpha_{\hat\mu} e^\beta_{\hat\nu}(\hat e^{\hat\sigma}_{\alpha,\beta} - \hat e^{\hat\sigma}_{\beta,\alpha})\hat\partial_{\hat\sigma} = C_{\hat\mu\hat\nu\cdot}^{\cdot\cdot\hat\sigma}\hat\partial_{\hat\sigma} \tag{4}$$

defining the Cartan coefficients which help us to write the *connection* components in local frames as

$$\hat\Gamma^{\hat\sigma}_{\hat\mu\hat\nu} = e^\alpha_{\hat\mu} e^\beta_{\hat\nu}(\hat e^{\hat\sigma}_\gamma \Gamma^\gamma_{\alpha\beta} - \hat e^{\hat\sigma}_{\beta,\alpha}) = \tfrac{1}{2}\eta^{\hat\sigma\hat\lambda}(C_{\hat\mu\hat\nu\hat\lambda} + C_{\hat\lambda\hat\mu\hat\nu} + C_{\hat\lambda\hat\nu\hat\mu}) \tag{5}$$

where $\Gamma^\gamma_{\alpha\beta}$ are the usual Christoffel symbols.

The Minkowski metric η remains invariant under the transformations of its *gauge* group, $G(\eta) = \mathrm{O}(3,1)$. This has as subgroup the Lorentz group, $L^\uparrow_+$, of the transformations $\Lambda[A(\omega)]$ corresponding to the transformations $A(\omega) \in SL(2,\mathbb{C})$ through the canonical homomorphism [8]. In the standard *covariant* parametrization, with the real parameters $\omega^{\hat\alpha\hat\beta} = -\omega^{\hat\beta\hat\alpha}$, we have

$$A(\omega) = \mathrm{e}^{-\frac{\mathrm{i}}{2}\omega^{\hat\alpha\hat\beta}S_{\hat\alpha\hat\beta}} \tag{6}$$

where $S_{\hat\alpha\hat\beta}$ are the covariant basis generators of the $sl(2,\mathbb{C})$ Lie algebra which satisfy

$$[S_{\hat\mu\hat\nu},\ S_{\hat\sigma\hat\tau}] = \mathrm{i}(\eta_{\hat\mu\hat\tau}\, S_{\hat\nu\hat\sigma} - \eta_{\hat\mu\hat\sigma}\, S_{\hat\nu\hat\tau} + \eta_{\hat\nu\hat\sigma}\, S_{\hat\mu\hat\tau} - \eta_{\hat\nu\hat\tau}\, S_{\hat\mu\hat\sigma}). \tag{7}$$

For small values of $\omega^{\hat\alpha\hat\beta}$ the matrix elements of the transformations Λ can be written as

$$\Lambda^{\hat\mu\,\cdot}_{\cdot\,\hat\nu}[A(\omega)] = \delta^{\hat\mu}_{\hat\nu} + \omega^{\hat\mu\,\cdot}_{\cdot\,\hat\nu} + \cdots. \tag{8}$$

Now we assume that M is orientable and time-orientable such that $L^{\uparrow}_{+}$ can be considered as the gauge group of the Minkowski metric [3]. Then the fields with spin can be defined as in the case of the flat spacetime, with the help of the finite-dimensional *linear* representations, ρ, of the $SL(2,\mathbb{C})$ group [8]. In general, the fields $\psi_\rho : M \to V_\rho$ are defined over M with values in the vector spaces V_ρ of the representations ρ. In what follows we systematically use the bases of V_ρ labelled only by spinor or vector *local* indices defined with respect to the axes of the local frames given by the tetrad fields. These will not be written explicitly, except when demanded by the concrete calculation needs.

The relativistic covariant field equations are derived from actions [1, 7]

$$\mathcal{S}[\psi_\rho, e] = \int \mathrm{d}^4x\sqrt{g}\,\mathcal{L}(\psi_\rho, D^{\rho}_{\hat\mu}\psi_\rho) \qquad g = |\det(g_{\mu\nu})| \tag{9}$$

depending on the matter fields, ψ_ρ, and the components of the tetrad fields, e, which represent the gravitational degrees of freedom. Recently, it was shown that this action can be completed by adding a term with the integration measure $\mathrm{d}^4x\,\Phi$ (instead of $\mathrm{d}^4x\sqrt{g}$) where Φ can be expressed in terms of scalar fields independent on e (or g) [18]. This new term allows one to define a new global scale symmetry which, in our opinion, is compatible with the geometric symmetries we study here. Therefore, without loss of generality, we can restrict ourselves to actions of the traditional form (9) in which the canonical variables are the components of the fields ψ_ρ and e.

The covariant derivatives,

$$D^{\rho}_{\hat\alpha} = e^{\mu}_{\hat\alpha} D^{\rho}_{\mu} = \hat\partial_{\hat\alpha} + \frac{\mathrm{i}}{2}\,\rho(S^{\hat\beta\,\cdot}_{\cdot\,\hat\gamma})\,\hat\Gamma^{\hat\gamma}_{\hat\alpha\hat\beta} \tag{10}$$

assure the covariance of the whole theory under the *tetrad gauge* transformations,

$$\begin{aligned}
\hat e^{\hat\alpha}_{\mu}(x) &\to \hat e'^{\hat\alpha}_{\mu}(x) = \Lambda^{\hat\alpha\,\cdot}_{\cdot\,\hat\beta}[A(x)]\,\hat e^{\hat\beta}_{\mu}(x)\\
e^{\mu}_{\hat\alpha}(x) &\to e'^{\mu}_{\hat\alpha}(x) = \Lambda^{\cdot\,\hat\beta}_{\hat\alpha\,\cdot}[A(x)]\,e^{\mu}_{\hat\beta}(x)\\
\psi_\rho(x) &\to \psi'_\rho(x) = \rho[A(x)]\,\psi_\rho(x)
\end{aligned} \tag{11}$$

determined by the mappings $A : M \to SL(2,\mathbb{C})$ the values of which are the local $SL(2,\mathbb{C})$ transformations $A(x) \equiv A[\omega(x)]$. These mappings can be organized as a group, $\mathcal{G}$, with respect to the multiplication $\times$ defined as $(A' \times A)(x) = A'(x)A(x)$. The notation Id stands for the mapping identity, $\mathrm{Id}(x) = 1 \in SL(2,\mathbb{C})$, while A^{-1} is the inverse of A, $(A^{-1})(x) = [A(x)]^{-1}$.

2.2. Combined transformations

The general coordinate transformations are automorphisms of M which, in the passive mode, can be seen as changes of the local charts corresponding to the same domain of M [3, 9]. If x and x' are the coordinates of a point in two different charts then there is a mapping

ϕ between these charts giving the coordinate transformation, $x \to x' = \phi(x)$. These transformations form a group with respect to the composition of mappings, $\circ$, defined as usual, i.e. $(\phi' \circ \phi)(x) = \phi'[\phi(x)]$. We denote this group by $\mathcal{A}$, its identity map by id and the inverse mapping of ϕ by ϕ^{-1}.

The automorphisms change all the components carrying natural indices including those of the tetrad fields [1] changing thus the positions of the local frames with respect to the natural ones. If we assume that the physical experiment makes reference to the axes of the local frame then situations can arise when several corrections of the positions of these frames are needed before (or after) a general coordinate transformation. Obviously, these have to be done with the help of a suitable gauge transformation associated to the automorphisms. Thus it is useful to introduce the *combined* transformations denoted by (A, ϕ) and defined as gauge transformations, given by $A \in \mathcal{G}$, followed by automorphisms, $\phi \in \mathcal{A}$. In this new notation the pure gauge transformations appear as (A,id) while the automorphisms are denoted by (Id, ϕ).

The effect of a combined transformation (A, ϕ) upon our basic fields, ψ_ρ, e and $\hat{e}$ is $x \to x' = \phi(x)$, $e(x) \to e'(x')$, $\hat{e}(x) \to \hat{e}'(x')$ and $\psi_\rho(x) \to \psi'_\rho(x') = \rho[A(x)]\psi_\rho(x)$ where e' are the transformed tetrads of the components

$$e'^{\mu}_{\hat{\alpha}}[\phi(x)] = \Lambda^{\cdot\hat{\beta}}_{\hat{\alpha}\cdot}[A(x)]\, e^{\nu}_{\hat{\beta}}(x) \frac{\partial \phi^\mu(x)}{\partial x^\nu} \tag{12}$$

while the components of $\hat{e}'$ have to be calculated according to equations (1). Thus we have written down the most general transformation laws that leave invariant the action in the sense that $\mathcal{S}[\psi'_\rho, e'] = \mathcal{S}[\psi_\rho, e]$. The field equations derived from $\mathcal{S}$, written in local frames as $(E_\rho\psi_\rho)(x) = 0$, *covariantly* transform according to the rule

$$(E_\rho\psi_\rho)(x) \to (E'_\rho\psi'_\rho)(x') = \rho[A(x)](E_\rho\psi_\rho)(x) \tag{13}$$

since the operators E_ρ involve covariant derivatives [1].

The association among the transformations of the groups $\mathcal{G}$ and $\mathcal{A}$ must lead to a new group with a specific multiplication. In order to explore the form of this new operation it is convenient to use the composition among the mappings A and ϕ (taken only in this order) giving new mappings, $A \circ \phi \in \mathcal{G}$, defined as $(A \circ \phi)(x) = A[\phi(x)]$. The calculation rules $\mathrm{Id} \circ \phi = \mathrm{Id}$, $A \circ \mathrm{id} = \mathrm{A}$ and $(A' \times A) \circ \phi = (A' \circ \phi) \times (A \circ \phi)$ are obvious. With these ingredients we define the new multiplication

$$(A', \phi') * (A, \phi) = ((A' \circ \phi) \times A, \phi' \circ \phi). \tag{14}$$

It is clear that now the identity is $(\mathrm{Id}, \mathrm{id})$ while the inverse of a pair (A, ϕ) reads

$$(A, \phi)^{-1} = (A^{-1} \circ \phi^{-1}, \phi^{-1}). \tag{15}$$

The operation $*$ is well defined and represents the composition among the combined transformations since these can be expressed, according to their definition, as $(A, \phi) = (\mathrm{Id}, \phi) * (\mathrm{A}, \mathrm{id})$. Furthermore, we can convince ourselves that if we perform successively two arbitrary combined transformations, (A, ϕ) and (A', ϕ'), then the resulting transformation is just $(A', \phi') * (A, \phi)$ as given by equation (14). This means that the combined transformations form a group with respect to the multiplication $*$. It is not difficult to verify that this group, denoted by $\tilde{\mathcal{G}}$, is the semidirect product $\tilde{\mathcal{G}} = \mathcal{G}Ⓢ\mathcal{A}$ where $\mathcal{G}$ is the *invariant* subgroup while $\mathcal{A}$ is a usual one.

In the theories involving only vector and tensor fields we do not need to use the combined transformations defined above since the theory is independent of the positions of the local frames. This can be easily shown even in our approach where we use field components with local indices. Indeed, if we perform a combined transformation (A, ϕ) then any tensor field of rank (p, q),

$$\psi^{\hat{\alpha}_1,\hat{\alpha}_2,\ldots,\hat{\alpha}_p}_{\hat{\beta}_1,\hat{\beta}_2,\ldots,\hat{\beta}_q} = \hat{e}^{\hat{\alpha}_1}_{\mu_1} \cdots \hat{e}^{\hat{\alpha}_p}_{\mu_p}\, e^{\nu_1}_{\hat{\beta}_1} \ldots e^{\nu_q}_{\hat{\beta}_q}\, \psi^{\mu_1,\mu_2,\ldots,\mu_p}_{\nu_1,\nu_2,\ldots,\nu_q} \tag{16}$$

transforms according to the representation

$$\rho^{\hat\beta_1,\hat\beta_2,\ldots,\hat\beta_q;\,\hat\alpha'_1,\hat\alpha'_2,\ldots,\hat\alpha'_p}_{\hat\alpha_1,\hat\alpha_2,\ldots,\hat\alpha_p;\,\hat\beta'_1,\hat\beta'_2,\ldots,\hat\beta'_q}(A) = \Lambda^{\hat\beta_1\cdot}_{\cdot\,\hat\beta'_1}(A)\ldots\ \Lambda^{\cdot\,\hat\alpha'_1}_{\hat\alpha_1\cdot}(A)\ldots \tag{17}$$

such that the resulting transformation law of the components carrying natural indices,

$$\psi'^{\mu_1,\ldots}_{\nu_1,\ldots}(x') = \frac{\partial x'^{\mu_1}}{\partial x^{\sigma_1}}\cdots\frac{\partial x^{\tau_1}}{\partial x'^{\nu_1}}\cdots\psi^{\sigma_1,\ldots}_{\tau_1,\ldots}(x) \tag{18}$$

is just the familiar one [1]. In other words, in this case the effect of the combined transformations reduces to that of their automorphisms. However, when the half-integer spin fields are involved this is no longer true and we must use the combined transformations of $\tilde{\mathcal{G}}$ if we want to keep the positions of the local frames under control.

3. External symmetry

In general, the symmetry of any manifold M is given by its isometry group whose transformations leave the metric tensor invariant in any chart. The scalar field transforms under isometries according to the standard scalar representation generated by the orbital generators related to the Killing vectors of M [1,3,9]. In what follows we propose a possible generalization of this theory of symmetry to the fields with spin, defining the external symmetry group and its representations.

3.1. Isometries

There are conjectures when several coordinate transformations, $x \to x' = \phi_\xi(x)$, depend on N independent real parameters, ξ^a $(a, b, c, \ldots = 1, 2, \ldots, N)$, such that $\xi = 0$ corresponds to the identity map, $\phi_{\xi=0} = \mathrm{id}$. These mappings form a Lie group [19] if they accomplish the composition rule

$$\phi_{\xi'}\circ\phi_\xi = \phi_{f(\xi',\xi)} \tag{19}$$

where the functions f define the group multiplication. These must satisfy $f^a(0,\xi) = f^a(\xi,0) = \xi^a$ and $f^a(\xi^{-1},\xi) = f^a(\xi,\xi^{-1}) = 0$ where ξ^{-1} are the parameters of the inverse mapping of ϕ_ξ, $\phi_{\xi^{-1}} = \phi_\xi^{-1}$. Moreover, the structure constants of this group can be calculated as [20]

$$c_{abc} = \left(\frac{\partial f^c(\xi,\xi')}{\partial\xi^a\partial\xi'^b} - \frac{\partial f^c(\xi,\xi')}{\partial\xi^b\partial\xi'^a}\right)\bigg|_{\xi=\xi'=0}. \tag{20}$$

For small values of the group parameters the infinitesimal transformations, $x^\mu \to x'^\mu = x^\mu + \xi^a k_a^\mu(x) + \cdots$, are given by the vectors k_a whose components,

$$k_a^\mu = \frac{\partial\phi_\xi^\mu}{\partial\xi^a}\bigg|_{\xi=0} \tag{21}$$

satisfy the identities

$$k_a^\mu k^\nu_{b,\mu} - k_b^\mu k^\nu_{a,\mu} + c_{abc}k_c^\nu = 0 \tag{22}$$

resulting from equations (19) and (20).

In what follows we restrict ourselves to considering only the *isometry* transformations, $x' = \phi_\xi(x)$, which leave invariant the components of the metric tensor [1,9], i.e.

$$g_{\alpha\beta}(x')\frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu} = g_{\mu\nu}(x). \tag{23}$$

These form the isometry group $I(M) \subset \mathcal{A}$ which is the Lie group giving the symmetry of the spacetime M. If this has N independent parameters then k_a, $a = 1, 2, \ldots, N$, are independent Killing vectors which satisfy $k_{a\,\mu;\nu} + k_{a\,\nu;\mu} = 0$. Their corresponding Lie derivatives form a basis of the Lie algebra $\mathrm{i}(M)$ of the group $I(M)$ [9].

However, in practice we are interested to find the operators of the relativistic quantum theory related to these geometric objects which describe the symmetry of the background. For this reason we focus upon the operator-valued representations [21] of the group $I(M)$ and its algebra. The *scalar* field $\psi : M \to \mathbb{C}$ transforms under isometries as $\psi(x) \to \psi'[\phi_\xi(x)] = \psi(x)$. This rule defines the representation $\phi_\xi \to T_\xi$ of the group $I(M)$ whose operators have the action $\psi' = T_\xi\psi = \psi \circ \phi_\xi^{-1}$. Thus, the operators of infinitesimal transformations, $T_\xi = 1 - \mathrm{i}\xi^a L_a + \cdots$, depend on the basis generators

$$L_a = -\mathrm{i}k_a^\mu \partial_\mu \qquad a = 1, 2, \ldots, N \tag{24}$$

which are completely determined by the Killing vectors. From equation (22) we see that they obey the commutation rules

$$[L_a, L_b] = \mathrm{i}c_{abc} L_c \tag{25}$$

given by the structure constants of $I(M)$. In other words they form a basis of the operator-valued representation of the Lie algebra $i(M)$ in a carrier space of scalar fields. Notice that in the usual quantum mechanics the operators similar to the generators L_a are called often *orbital* generators.

3.2. The group of external symmetry

The problem is how the whole geometric framework of the theories with spin may be transformed under isometries where we explicitly use the local frames. Since the isometry is a general coordinate transformation it changes the relative positions of the local frames with respect to the natural ones. This fact may be an impediment when one intends to study the symmetries of the theories with spin in local frames. For this reason it is natural to suppose that the good symmetry transformations we need are isometries preceded by appropriate gauge transformations which should assure that not only the form of the metric tensor are conserved but the form of the tetrad field components too. However, these transformations are nothing other than *particular* combined transformations whose automorphisms are isometries.

Thus we arrive at the main point of our proposal. We introduce the *external symmetry* transformations, (A_ξ, ϕ_ξ), as particular combined transformations involving isometries, $\phi_\xi \in I(M)$, and corresponding gauge transformations, $A_\xi \in \mathcal{G}$, necessary to *preserve the gauge*. We assume that in a fixed gauge, given by the tetrad fields e and $\hat{e}$, A_ξ is defined by

$$\Lambda^{\hat{\alpha}}_{\cdot\,\hat{\beta}}[A_\xi(x)] = \hat{e}^{\hat{\alpha}}_\mu[\phi_\xi(x)] \frac{\partial \phi^\mu_\xi(x)}{\partial x^\nu} e^\nu_{\hat{\beta}}(x) \tag{26}$$

with the supplementary condition $A_{\xi=0}(x) = 1 \in SL(2, \mathbb{C})$. Since ϕ_ξ is an isometry, equation (23) guarantees that $\Lambda[A_\xi(x)] \in L^\uparrow_+$ and, implicitly, $A_\xi(x) \in SL(2, \mathbb{C})$. Then the transformation laws of our fields are

$$(A_\xi, \phi_\xi): \quad \begin{aligned} & x \to x' = \phi_\xi(x) \\ & e(x) \to e'(x') = e[\phi_\xi(x)] \\ & \hat{e}(x) \to \hat{e}'(x') = \hat{e}[\phi_\xi(x)] \\ & \psi_\rho(x) \to \psi'_\rho(x') = \rho[A_\xi(x)]\psi_\rho(x). \end{aligned} \tag{27}$$

The main virtue of these transformations is that they leave *invariant* the form of the operators of the field equations, E_ρ, in local frames. This is because the components of the tetrad fields and, consequently, the covariant derivatives in local frames, $D^\rho_{\hat{\alpha}}$, do not change their form.

For small ξ^a the covariant $SL(2,\mathbb{C})$ parameters of $A_\xi(x) \equiv A[\omega_\xi(x)]$ can be written as $\omega_\xi^{\hat\alpha\hat\beta}(x) = \xi^a \Omega_a^{\hat\alpha\hat\beta}(x) + \cdots$ where, according to equations (6), (8) and (26), we have

$$\Omega_a^{\hat\alpha\hat\beta} \equiv \left.\frac{\partial\omega_\xi^{\hat\alpha\hat\beta}}{\partial\xi^a}\right|_{\xi=0} = (\hat e_\mu^{\hat\alpha}\, k_{a,\nu}^\mu + \hat e_{\nu,\mu}^{\hat\alpha} k_a^\mu) e_{\hat\lambda}^\nu \eta^{\hat\lambda\hat\beta}. \tag{28}$$

We must specify that these functions are antisymmetric if and only if k_a are Killing vectors. This indicates that the association among isometries and the gauge transformations defined by equation (26) is correct.

It remains to show that the transformations (A_ξ, ϕ_ξ) form a Lie group related to $I(M)$. Starting with equation (26) after a little calculation we find that

$$(A_{\xi'} \circ \phi_\xi) \times A_\xi = A_{f(\xi',\xi)} \tag{29}$$

and, according to equations (14) and (19), we obtain

$$(A_{\xi'}, \phi_{\xi'}) * (A_\xi, \phi_\xi) = (A_{f(\xi',\xi)}, \phi_{f(\xi',\xi)}) \tag{30}$$

and $(A_{\xi=0}, \phi_{\xi=0}) = (\mathrm{Id}, \mathrm{id})$. Thus we have shown that the pairs (A_ξ, ϕ_ξ) form a Lie group with respect to the operation $*$. We say that this is the external symmetry group of M and we denote it by $S(M) \subset \tilde{\mathcal{G}}$. From equation (30) we understand that $S(M)$ is *locally isomorphic* with $I(M)$ and, therefore, the Lie algebra of $S(M)$, denoted by $s(M)$, is isomorphic with $i(M)$ having the same structure constants. In our opinion, $S(M)$ must be isomorphic with the universal covering group of $I(M)$ since it has the topology induced by $SL(2,\mathbb{C})$ which is simply connected. In general, the number of group parameters of $I(M)$ or $S(M)$ (which is equal to the number of the independent Killing vectors of M) can be $0 \leqslant N \leqslant 10$.

The form of the external symmetry transformations is strongly dependent on the choice of the local chart as well as that of the tetrad gauge. If we change simultaneously the gauge and the coordinates with the help of a combined transformation (A, ϕ) then each $(A_\xi, \phi_\xi) \in S(M)$ transforms as

$$(A_\xi, \phi_\xi) \to (A'_\xi, \phi'_\xi) = (A, \phi) * (A_\xi, \phi_\xi) * (A, \phi)^{-1} \tag{31}$$

which means that

$$A'_\xi = \{[(A \circ \phi_\xi) \times A_\xi] \times A^{-1}\} \circ \phi^{-1} \tag{32}$$

$$\phi'_\xi = (\phi \circ \phi_\xi) \circ \phi^{-1}. \tag{33}$$

Obviously, these transformations define automorphisms of $S(M)$.

3.3. Representations

The last of equations (27) which gives the transformation law of the field ψ_ρ defines the operator-valued representation $(A_\xi, \phi_\xi) \to T_\xi^\rho$ of the group $S(M)$,

$$(T_\xi^\rho \psi_\rho)[\phi_\xi(x)] = \rho[A_\xi(x)]\psi_\rho(x) \tag{34}$$

which leaves invariant the operator of the field equation in local frames obeying

$$T_\xi^\rho E_\rho (T_\xi^\rho)^{-1} = E_\rho. \tag{35}$$

Since $A_\xi(x) \in SL(2,\mathbb{C})$ we say that this representation is *induced* by the representation ρ of $SL(2,\mathbb{C})$ [21, 22]. As we have shown in section 2.2, if ρ is a vector or tensor representation (having only integer spin components) then the effect of the transformation (34) upon the components carrying natural indices is due only to ϕ_ξ. However, for the representations with half-integer spin the presence of A_ξ is crucial since there are no natural indices. In addition, this allows us to define the generators of the representations (34) for any spin.

The basis generators of the representations of the Lie algebra $s(M)$ are the operators

$$X_a^\rho = \mathrm{i}\, \frac{\partial T_\xi^\rho}{\partial \xi^a}\bigg|_{\xi=0} = L_a + S_a^\rho \tag{36}$$

which appear as sums among the orbital generators defined by equation (24) and the *spin terms* which have the action

$$(S_a^\rho \psi_\rho)(x) = \rho[S_a(x)]\psi_\rho(x). \tag{37}$$

This is determined by the form of the *local* $sl(2, \mathbb{C})$ generators,

$$S_a(x) = \mathrm{i}\, \frac{\partial A_\xi(x)}{\partial \xi^a}\bigg|_{\xi=0} = \frac{1}{2}\, \Omega_a^{\hat\alpha\hat\beta}(x) S_{\hat\alpha\hat\beta} \tag{38}$$

that depend on the functions (28). Furthermore, if we derive equation (29) with respect to ξ and ξ' then from equations (8), (20) and (28), after a few manipulations, we obtain the identities

$$\eta_{\hat\alpha\hat\beta}(\Omega_a^{\hat\alpha\hat\mu}\Omega_b^{\hat\beta\hat\nu} - \Omega_b^{\hat\alpha\hat\mu}\Omega_a^{\hat\beta\hat\nu}) + k_a^\mu \Omega_{b,\mu}^{\hat\mu\hat\nu} - k_b^\mu \Omega_{a,\mu}^{\hat\mu\hat\nu} + c_{abc}\Omega_c^{\hat\mu\hat\nu} = 0. \tag{39}$$

Thus we have

$$[S_a^\rho, S_b^\rho] + [L_a, S_b^\rho] - [L_b, S_a^\rho] = \mathrm{i}c_{\mathrm{abc}}\mathrm{S}_{\mathrm{c}}^\rho \tag{40}$$

and, according to equation (25), we find the expected commutation rules

$$[X_a^\rho, X_b^\rho] = \mathrm{i}c_{abc} X_c^\rho. \tag{41}$$

Thus, we have derived the basis generators of the operator-valued representation of $s(M)$ induced by the linear representation ρ of $sl(2, \mathbb{C})$. All the operators of this representation commute with the operator E_ρ since, according to equations (35) and (36), we have

$$[E_\rho, X_a^\rho] = 0 \qquad a = 1, 2, \ldots, N. \tag{42}$$

Therefore, for defining quantum modes we can use the set of commuting operators containing the Casimir operators of $s(M)$, the operators of its Cartan subalgebra and E_ρ.

Finally, we must specify that the basis generators (36) of the representations of the $s(M)$ algebra can be written in covariant form as

$$X_a^\rho = -\mathrm{i}k_a^\mu D_\mu^\rho + \tfrac{1}{2}\, k_{a\,\mu;\nu}\, e_{\hat\alpha}^\mu\, e_{\hat\beta}^\nu\, \rho(S^{\hat\alpha\hat\beta}) \tag{43}$$

generalizing thus the important result obtained in [11] for the Dirac field.

3.4. Manifest covariance

The action of the operators X_a^ρ depends on the choice of many elements: the natural coordinates, the tetrad gauge, the group parametrization and the representation ρ. What is important here is that they are strongly dependent on the tetrad gauge fixing even in the case of the representations with integer spin. This is because the covariant parametrization of the $SL(2, \mathbb{C})$ group is defined with respect to the axes of the local frames. In general, if we consider the representation $(A_\xi, \phi_\xi) \to T_\xi^\rho$ and we perform the transformation (31) then we derive the *equivalent* representation, $(A'_\xi, \phi'_\xi) \to T_\xi'^\rho$. Its generators calculated from equations (32) indicate that in this case the equivalence relations are much more complicated than those of the usual theory of linear representations. Without entering into other technical details we specify that if we change only the gauge with the help of the transformation (A, id) then the local $sl(2,\mathbb{C})$ generators (38) transform as

$$S_a(x) \to S'_a(x) = A(x)S_a(x)A(x)^{-1} + k_a^{\sigma}(x)\Lambda_{\hat{\alpha}\hat{\mu},\sigma}[A(x)]\Lambda^{\cdot\hat{\mu}}_{\hat{\beta}\cdot}[A(x)]S^{\hat{\alpha}\hat{\beta}} \quad (44)$$

while the orbital parts do not change their form. This means that the gauge transformations change, in addition, the commutation relations among the spin and orbital parts of the generators X_a^{ρ}.

The consequence is that we can find gauge fixings where the local $sl(2,\mathbb{C})$ generators $S_a(x)$, $a = 1, 2, \ldots, n$ $(n \leqslant N)$, corresponding to a subgroup $H \subset S(M)$, are independent on x and, therefore, $[S_a^{\rho}, L_b] = 0$ for all $a = 1, 2, \ldots, n$ and $b = 1, 2, \ldots, N$. Then the operators S_a^{ρ}, $a = 1, 2, \ldots, n$ are just the basis generators of an usual linear representation of H and the field ψ_ρ behaves *manifestly covariant* under the external symmetry transformations of this subgroup. Of course, when $H = S(M)$ we say simply that the field ψ_ρ is manifest covariant.

The simplest examples are the manifest covariant fields of special relativity. Since here the spacetime M is flat, the metric in Cartesian coordinates is $g_{\mu\nu} = \eta_{\mu\nu}$ and one can use the *inertial* (local) frames with $e^{\mu}_{\nu} = \hat{e}^{\mu}_{\nu} = \delta^{\mu}_{\nu}$. Then the isometries are just the transformations $x' = \Lambda[A(\omega)]x - a$ of the Poincaré group, $\mathcal{P}_+^{\uparrow} = T(4)\circledS L_+^{\uparrow}$ [8]. If we denote by $\xi^{(\mu\nu)} = \omega^{\mu\nu}$ the $SL(2,\mathbb{C})$ parameters and by $\xi^{(\mu)} = a^{\mu}$ those of the translation group $T(4)$, then it is a simple exercise to calculate the basis generators

$$X^{\rho}_{(\mu)} = \mathrm{i}\partial_\mu \quad (45)$$

$$X^{\rho}_{(\mu\nu)} = \mathrm{i}(\eta_{\mu\alpha}x^{\alpha}\partial_\nu - \eta_{\nu\alpha}x^{\alpha}\partial_\mu) + \rho(S_{\mu\nu}) \quad (46)$$

which show us that ψ_ρ transforms manifestly covariant. On the other hand, it is clear that the group $S(M) \equiv \tilde{\mathcal{P}}_+^{\uparrow} = T(4)\circledS SL(2,\mathbb{C})$ is just the universal covering group of $I(M) \equiv \mathcal{P}_+^{\uparrow}$.

In general, there are many cases of curved spacetimes for which one can choose suitable local frames allowing one to introduce manifest covariant fields with respect to a subgroup $H \subset S(M)$ or even the whole group $S(M)$. In our opinion, this is possible only when H or $S(M)$ are at most subgroups of $\tilde{\mathcal{P}}_+^{\uparrow}$.

4. The central symmetry

Let us take as a first example the spacetimes M which have spherically symmetric static chart that will be referred to here as *central* charts. These manifolds have the isometry group $I(M) = T(1) \otimes SO(3)$ of time translations and space rotations.

4.1. Central charts

In a central chart with Cartesian coordinates $x^0 = t$ and x^i $(i, j, k, \ldots = 1, 2, 3)$, the metric tensor is time independent and transforms manifestly covariant under the rotations $R \in SO(3)$ of the space coordinates,

$$t' = t \qquad x'^i = R^{i}_{\cdot j}(\omega)x^j = x^i + \omega^{i}_{\cdot j}x^j + \cdots \quad (47)$$

denoted simply by $x \to x' = Rx$. Here the most general form of the line element,

$$\mathrm{d}s^2 = g_{\mu\nu}(x)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = A(r)\,\mathrm{d}t^2 - [B(r)\delta_{ij} + C(r)x^i x^j]\,\mathrm{d}x^i\,\mathrm{d}x^j \quad (48)$$

may involve three functions, A, B and C, depending only on the Euclidean norm of $\vec{x}$, $r = |\vec{x}|$. In applications it is convenient to replace these functions by new ones, u, v and w, defined as

$$A = w^2 \qquad B = \frac{w^2}{v^2} \qquad C = \frac{1}{r^2}\left(\frac{w^2}{u^2} - \frac{w^2}{v^2}\right). \quad (49)$$

Other useful central charts are those with spherical coordinates, r, θ, ϕ, commonly associated with the Cartesian space ones. Here the line elements are

$$\mathrm{d}s^2 = w^2\,\mathrm{d}t^2 - \frac{w^2}{u^2}\,\mathrm{d}r^2 - \frac{w^2}{v^2}r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2). \tag{50}$$

In these charts we see that the advantage of the new functions we have introduced is of simple transformation laws under the isotropic dilations which change only the radial coordinate, $r \to r'(r)$, without affecting the central symmetry of the line element. These transformations,

$$u'(r') = u(r)\left|\frac{\mathrm{d}r'(r)}{\mathrm{d}r}\right| \qquad v'(r') = v(r)\frac{r'(r)}{r} \qquad w'(r') = w(r) \tag{51}$$

allow one to choose desired forms for the functions u, v and w.

4.2. The Cartesian gauge

The Cartesian gauge in central charts was mentioned some time ago [15] but is little used in concrete problems since it leads to complicated calculations in spherical coordinates. However, in Cartesian coordinates this gauge has the advantage of explicitly pointing out the global central symmetry of the manifold. In [12] we proposed a version of Cartesian gauge in central charts with Cartesian coordinates that preserve the manifest covariance under rotations (47) in the sense that the 1-forms $\mathrm{d}\hat{x}^{\hat\mu} = \hat{e}^{\hat\mu}_{\alpha}(x)\,\mathrm{d}x^\alpha$ transform as

$$\mathrm{d}\hat{x}^{\hat\mu} \to \mathrm{d}\hat{x}'^{\hat\mu} = \hat{e}^{\hat\mu}_{\alpha}(x')\,\mathrm{d}x'^\alpha = (R\,\mathrm{d}\hat{x})^{\hat\mu}. \tag{52}$$

If the line element has the form (48) then the most general choice of the tetrad fields with the above property is

$$\hat{e}^0_0 = \hat{a}(r) \qquad \hat{e}^0_i = \hat{e}^i_0 = 0 \qquad \hat{e}^i_j = \hat{b}(r)\delta_{ij} + \hat{c}(r)x^i x^j + \hat{d}(r)\varepsilon_{ijk}x^k \tag{53}$$

$$e^0_0 = a(r) \qquad e^0_i = e^i_0 = 0 \qquad e^i_j = b(r)\delta_{ij} + c(r)x^i x^j + d(r)\varepsilon_{ijk}x^k \tag{54}$$

where, according to (3), (48) and (49), we must have

$$\hat{a} = w \qquad \hat{b} = \frac{w}{v}\cos\alpha \qquad \hat{c} = \frac{1}{r^2}\left(\frac{w}{u} - \frac{w}{v}\cos\alpha\right) \qquad \hat{d} = \frac{1}{r}\frac{w}{v}\sin\alpha \tag{55}$$

$$a = \frac{1}{w} \qquad b = \frac{v}{w}\cos\alpha \qquad c = \frac{1}{r^2}\left(\frac{u}{w} - \frac{v}{w}\cos\alpha\right) \qquad d = -\frac{1}{r}\frac{v}{w}\sin\alpha. \tag{56}$$

The angle α is an arbitrary function of r which is not explicitly involved in the expression of the metric tensor since it represents the angle of an arbitrary rotation of the local frame around the direction of $\vec{x}$, that does not change the relative position of $\vec{x}$ with respect to this frame.

When one defines the metric tensor such that $g_{\mu\nu|r=0} = \eta_{\mu\nu}$ then we have $u(0)^2 = v(0)^2 = w(0)^2 = 1$ and it is natural to take $\alpha(0) = 0$. On the other hand, from equations (55) and (56) we see that the function w must be positively defined in order to keep the same sense for the time axes of the natural and local frames. In addition, it is convenient to consider that the function u is positively defined too. However, the function $v = \eta_{\mathrm{P}}|v|$ has the sign given by the relative parity η_{P} which takes the value $\eta_{\mathrm{P}} = 1$ when the space axes of the local frame at $x = 0$ are parallel with those of the natural frame, and $\eta_{\mathrm{P}} = -1$ if these are antiparallel.

Now we have all the elements we need to calculate the generators of the representations T^ρ of the group $S(M)$. If we denote by $\xi^{(0)}$ the parameter of the time translations and by $\xi^{(i)} = \varepsilon_{ijk}\omega^{jk}/2$ the parameters of the rotations (47), we find that the local $sl(2,\mathbb{C})$ generators of equation (38) are just the $su(2)$ ones, i.e. $S_{(i)}(x) = S_i = \varepsilon_{ijk}S_{jk}/2$, such that the basis generators read

$$X^\rho_{(0)} = \mathrm{i}\partial_t \qquad X^\rho_{(i)} = L_{(i)} + \rho(S_i) \tag{57}$$

where $L_{(i)} = -\mathrm{i}\varepsilon_{ijk}x^j\partial_k$ are the usual components of the orbital angular momentum. Thus we obtain that the group $S(M) = T(1) \otimes SU(2)$ is the universal covering group of $I(M)$. Its transformations are gauge transformations $A_{\bar{\xi}} \in SU(2)$, independent of x, combined with the isometries of $I(M)$ given by $x \to x' = R(A_{\bar{\xi}})\,x$ and $t \to t' = t - \xi^{(0)}$. This means that, in this gauge, the field ψ_ρ transforms manifestly covariant. Moreover, the physical significance of the basis generators is the usual one, namely $X^\rho_{(0)}$ is the Hamiltonian operator while $X^\rho_{(i)} \equiv J^\rho_{(i)}$ are the components of the whole angular momentum operator.

We conclude that, in our Cartesian gauge, the local frames play the same role as the usual Cartesian rest frames of the central sources in flat spacetime since their axes are just those of projections of the angular momenta.

4.3. The diagonal gauge

In other gauge fixings the basis generators are quite different. A tetrad gauge widely used in central charts with spherical coordinates is the diagonal gauge defined by the 1-forms [16]

$$\mathrm{d}\hat{x}^0_s = w\,\mathrm{d}t \qquad \mathrm{d}\hat{x}^1_s = \frac{w}{u}\,\mathrm{d}r \qquad \mathrm{d}\hat{x}^2_s = r\frac{w}{v}\,\mathrm{d}\theta \qquad \mathrm{d}\hat{x}^3_s = r\frac{w}{v}\sin\theta\,\mathrm{d}\phi. \tag{58}$$

In this gauge the angular momentum operators of the canonical basis (where $J_{(\pm)} = J_{(1)} \pm \mathrm{i}J_{(2)}$) are [10]

$$J^\rho_{(\pm)} = L_{(\pm)} + \frac{\mathrm{e}^{\pm\mathrm{i}\phi}}{\sin\theta}\,\rho(S_{23}) \qquad J^\rho_{(3)} = L_{(3)}. \tag{59}$$

Thus one obtains a representation of $SU(2)$ where the spin terms do not commute with the orbital ones and, therefore, the field ψ_ρ does not transform manifestly covariant under rotations. In this case we can say that the spin part of the central symmetry remains partially hidden because of the diagonal gauge which determines special positions of the local frames with respect to the natural one. However, when this is an impediment one can change this gauge into the Cartesian one at any time by using a simple local rotation. For the flat spacetimes these transformations and their effects upon the Dirac equation are studied in [23]. Note that the form of the spin generators as well as that of the mentioned rotation depend on the enumeration of the 1-forms (58).

5. The dS and AdS symmetries

The backgrounds with highest external symmetry are the dS and the AdS spacetimes. We shall briefly discuss both these manifolds, denoted by M_ϵ where $\epsilon = 1$ for the dS case and $\epsilon = -1$ for the AdS one. Our goal here is to calculate the generators of the representations of the group $S(M_\epsilon)$ induced by those of $SL(2, \mathbb{C})$.

The dS and AdS spacetimes are hyperboloids in the (4 + 1)- or (3 + 2)-dimensional flat spacetimes, M^5_ϵ, of coordinates Z^A, $A, B, \ldots = 0, 1, 2, 3, 5$, and the metric $\eta(\epsilon) = \mathrm{diag}(1, -1, -1, -1, -\epsilon)$. The equation of the hyperboloid of radius $r_0 = 1/\hat{\omega}$ reads

$$-\eta_{AB}(\epsilon)Z^A Z^B = \epsilon\, {r_0}^2. \tag{60}$$

From their definitions it results that the dS or AdS spacetimes are homogeneous spaces of the pseudo-orthogonal groups $SO(4, 1)$ or $SO(3, 2)$ which play the role of gauge groups of the metric $\eta(\epsilon)$ (for $\epsilon = 1$ and -1 respectively) and represent just the isometry groups of these manifolds, $G[\eta(\epsilon)] = I(M_\epsilon)$. Then it is natural to use the *covariant* real parameters $\omega^{AB} = -\omega^{BA}$ since in this parametrization the orbital basis generators of the representations of $G[\eta(\epsilon)]$, carried by the spaces of functions over M^5_ϵ, have the usual form

$$L^5_{AB} = \mathrm{i}[\eta_{AC}(\epsilon)Z^C\partial_B - \eta_{BC}(\epsilon)Z^C\partial_A]. \tag{61}$$

They will give us directly the orbital basis generators of the representations of $S(M_\epsilon)$ in the carrier spaces of the functions defined over dS or AdS spacetimes.

5.1. Central charts

The hyperboloid equation can be solved in Cartesian dS/AdS coordinates, $x^0 = t$ and x^i ($i = 1, 2, 3$), which satisfy

$$\begin{aligned} Z^5 &= \hat{\omega}^{-1}\chi_\epsilon(r)\begin{cases}\cosh\hat{\omega}t & \text{if} \quad \epsilon = 1\\ \cos\hat{\omega}t & \text{if} \quad \epsilon = -1\end{cases}\\ Z^0 &= \hat{\omega}^{-1}\chi_\epsilon(r)\begin{cases}\sinh\hat{\omega}t & \text{if} \quad \epsilon = 1\\ \sin\hat{\omega}t & \text{if} \quad \epsilon = -1\end{cases}\\ Z^i &= x^i \end{aligned} \tag{62}$$

where we have denoted $\chi_\epsilon(r) = \sqrt{1 - \epsilon\,\omega^2 r^2}$. The line elements

$$\begin{aligned} \mathrm{d}s^2 &= \eta_{AB}(\epsilon)\,\mathrm{d}Z^A\,\mathrm{d}Z^B\\ &= \chi_\epsilon(r)^2\,\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{\chi_\epsilon(r)^2} - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \end{aligned} \tag{63}$$

are defined on the radial domains $D_r = [0, 1/\hat{\omega})$ or $D_r = [0, \infty)$ for dS or AdS respectively.

We calculate the Killing vectors and the orbital generators of the external symmetry in the Cartesian coordinates defined by equation (62) and the mentioned parametrization of $I(M_\epsilon)$ starting with the identification $\xi^{(AB)} = \omega^{AB}$. Then, from equations (24) and (61), after a little calculation, we obtain the orbital basis generators

$$L_{(05)} = \frac{\mathrm{i}\epsilon}{\hat{\omega}}\,\partial_t \tag{64}$$

$$L_{(j5)} = \frac{\mathrm{i}\epsilon}{\hat{\omega}}\chi_\epsilon(r)\begin{pmatrix}\cosh\hat{\omega}t\\ \cos\hat{\omega}t\end{pmatrix}\partial_j + \frac{\mathrm{i}x^j}{\chi_\epsilon(r)}\begin{pmatrix}\sinh\hat{\omega}t\\ \sin\hat{\omega}t\end{pmatrix}\partial_t \tag{65}$$

$$L_{(0j)} = \frac{\mathrm{i}}{\hat{\omega}}\chi_\epsilon(r)\begin{pmatrix}\sinh\hat{\omega}t\\ \sin\hat{\omega}t\end{pmatrix}\partial_j + \frac{\mathrm{i}x^j}{\chi_\epsilon(r)}\begin{pmatrix}\cosh\hat{\omega}t\\ \cos\hat{\omega}t\end{pmatrix}\partial_t \tag{66}$$

$$L_{(ij)} = -\mathrm{i}\,(x^i\,\partial_j - x^j\,\partial_i). \tag{67}$$

Furthermore, we consider the Cartesian tetrad gauge defined by equations (53)–(56) where, according to equation (63), we have

$$u(r) = \chi_\epsilon(r)^2 \qquad v(r) = w(r) = \chi_\epsilon(r). \tag{68}$$

In addition we take $\alpha = 0$. In this gauge we obtain the following local $sl(2, \mathbb{C})$ generators

$$S_{(05)}(x) = 0 \tag{69}$$

$$S_{(j5)}(x) = S_{0j}\begin{pmatrix}\sinh\hat{\omega}t\\ \sin\hat{\omega}t\end{pmatrix} + \frac{1}{r^2}[\chi_\epsilon(r) - 1]\left[\epsilon\frac{S_{jk}x^k}{\hat{\omega}}\begin{pmatrix}\cosh\hat{\omega}t\\ \cos\hat{\omega}t\end{pmatrix} - \frac{S_{0k}x^k x^j}{\chi_\epsilon(r)}\begin{pmatrix}\sinh\hat{\omega}t\\ \sin\hat{\omega}t\end{pmatrix}\right] \tag{70}$$

$$S_{(0j)}(x) = S_{0j}\begin{pmatrix}\cosh\hat{\omega}t\\ \cos\hat{\omega}t\end{pmatrix} + \frac{1}{r^2}[\chi_\epsilon(r) - 1]\left[\frac{S_{jk}x^k}{\hat{\omega}}\begin{pmatrix}\sinh\hat{\omega}t\\ \sin\hat{\omega}t\end{pmatrix} - \frac{S_{0k}x^k x^j}{\chi_\epsilon(r)}\begin{pmatrix}\cosh\hat{\omega}t\\ \cos\hat{\omega}t\end{pmatrix}\right] \tag{71}$$

$$S_{(ij)}(x) = S_{ij}. \tag{72}$$

With their help we can write the action of the spin terms (37) and, implicitly, that of the basis generators $X^\rho_{(AB)} = L_{(AB)} + S^\rho_{(AB)}$ of the representations of $S(M_\epsilon)$ induced by the representations ρ of $SL(2,\mathbb{C})$. Thus, it is not difficult to show that $S(M_\epsilon)$ is isomorphic with the universal covering group of $I(M_\epsilon)$ which in both cases ($\epsilon = \pm 1$) is a subgroup of the $SU(2,2)$ group. As expected, in central charts and Cartesian gauge the fields transform manifestly covariant only under the transformations of the subgroup $SU(2) \subset S(M_\epsilon)$.

5.2. Minkowskian charts

Another possibility is to solve the hyperboloid equation (60) in Minkowskian charts [1] where the coordinates, x^μ, are defined by

$$Z^5 = \hat{\omega}^{-1}\tilde{\chi}_\epsilon(s) \qquad Z^\mu = x^\mu \tag{73}$$

with $\tilde{\chi}_\epsilon(s) = \sqrt{1+\epsilon\,\hat{\omega}^2 s^2}$ and $s^2 = \eta_{\mu\nu}x^\mu x^\nu$. In these coordinates it is convenient to identify the hat indices with the usual ones and to not raise or lower these indices. Then we find that the metric tensor,

$$g_{\mu\nu}(x) = \eta_{\mu\nu} - \frac{\epsilon\,\hat{\omega}^2}{\tilde{\chi}_\epsilon(s)^2}\eta_{\mu\alpha}x^\alpha\eta_{\nu\beta}x^\beta \tag{74}$$

transforms manifestly covariant under the global $L^\uparrow_+$ transformations, $x'^\mu \to x^\mu = \Lambda^{\mu\cdot}_{\cdot\nu}x^\nu$. Moreover, the whole theory remains manifest covariant if we use the tetrad fields in the Lorentz gauge defined as [17]

$$e^\mu_\nu(x) = \delta^\mu_\nu + h_\epsilon(s)\eta_{\nu\alpha}x^\alpha x^\mu \qquad \hat{e}^\mu_\nu(x) = \delta^\mu_\nu + \hat{h}_\epsilon(s)\eta_{\nu\alpha}x^\alpha x^\mu \tag{75}$$

where

$$h_\epsilon(s) = \frac{1}{s^2}[\tilde{\chi}_\epsilon(s) - 1] \qquad \hat{h}_\epsilon(s) = \frac{1}{s^2}\left[\frac{1}{\tilde{\chi}_\epsilon(s)} - 1\right]. \tag{76}$$

First we calculate the $SO(4,1)$ or $SO(3,2)$ orbital generators,

$$L_{(\mu 5)} = \frac{\mathrm{i}\epsilon}{\hat{\omega}}\tilde{\chi}_\epsilon(s)\partial_\mu \tag{77}$$

$$L_{(\mu\nu)} = \mathrm{i}(\eta_{\mu\alpha}x^\alpha\partial_\nu - \eta_{\nu\alpha}x^\alpha\partial_\mu) \tag{78}$$

which are independent on the gauge fixing. We observe that in Minkowskian charts ∂_t is no more a Killing vector field as in the case of the central ones. However, here we have another advantage, namely that of the Lorentz gauge in which the local $sl(2,\mathbb{C})$ generators of equation (37) have the form

$$S_{(\mu 5)}(x) = -\frac{\epsilon}{\hat{\omega}s^2}[\tilde{\chi}_\epsilon(s) - 1]S_{\mu\alpha}x^\alpha \tag{79}$$

$$S_{(\mu\nu)}(x) = S_{\mu\nu} \tag{80}$$

showing that the field ψ_ρ transforms manifestly covariant under the whole $SL(2,\mathbb{C})$ subgroup of $S(M_\epsilon)$. Since these representations are induced just by those of $SL(2,\mathbb{C})$ we can say that in this gauge the manifest covariance is maximal.

6. Concluding remarks

The external symmetry in general relativity discussed here is the natural generalization of the Poincaré covariance of special relativity to curved spacetimes with isometries. When these exist, one can define the group $S(M)$ and its operator-valued representations carried by spaces of fields with arbitrary spin. In general, these representations are not manifestly covariant since they are induced by the linear representations of the $SL(2,\mathbb{C})$ group which is independent on the concrete structure of $S(M)$. In addition, their form is strongly dependent on the tetrad fields which seem to play here a similar role to the boosts in the theory of the Wigner-induced representations of the $\tilde{\mathcal{P}}_{+}^{\uparrow}$ group [8]. In any event, these representations are new mathematical objects the study of which could be interesting, especially as regards their classification.

From the physical point of view, the systems with isometries can be free fields defined on backgrounds with given symmetry or interacting matter fields coupled to a gravitational field with symmetric boundary conditions. The Killing vectors of these systems give us the basis generators of the operator-valued representations of the $s(M)$ algebra which commute with the operators of the covariant field equations. We obtain thus a collection of conserved observables which offers us the possibility to choose suitable sets of commuting operators which should determine the quantum states. On the other hand, since the concrete form of these generators depends on the choice of both the natural and local frames, the commutation rules among their spin and orbital parts are determined by the tetrad gauge. Consequently, the results of the local measurement of the spin observables may depend on the choice of the local frames. This suggests that it should be interesting to investigate new spin effects in different charts and tetrad gauge fixings.

In the general case of interacting matter fields coupled with gravity one can find solutions without isometries for which our theory of external symmetry does not make sense. Of course, we remain with the combined transformations of the group $\tilde{\mathcal{G}}$ but, as mentioned, these are not able to produce specific conserved magnitudes (apart from the stress-energy tensor of the matter fields [1]). Therefore, if we want to find new invariants we are forced to look for generalizations of the external symmetry transformations to the systems without isometries. In our opinion, these may be transformations of a well defined Lie group which should leave invariant the form of the field equations in local frames. These could be new generalized combined transformations embedding geometric, gauge and even internal symmetry transformations in reasonable physical limits, as those fixed by the no-go theorem in the flat spacetime. Thus we get a new complicated and sensitive problem that may be considered in the context of the actual theories with large number of extra dimensions and high symmetries or supersymmetries.

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